Adiabatic motion of a quantum particle in a two-dimensional magnetic field

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Abstract. The adiabatic motion of a charged, spinning, quantum particle in a two-dimensional (i.e. of constant direction) magnetic field is studied. A suitable set of operators generalizing the kinematical momenta and the guiding centre operators of a particle moving in a homogeneous magnetic field is constructed. This allows us to separate the two degrees of freedom of the system into a *fast* and a *slow* one which are, in the classical limit, the rapid rotation of the particle around the guiding centre and the slow guiding centre drift. In terms of these operators the Hamiltonian of the system can be rewritten as a power series in the magnetic length $l_B = \sqrt{\hbar c/eB}$, and the fast and slow dynamics separates. The effective guiding centre Hamiltonian is obtained to second order in the adiabatic parameter l_B and reproduces correctly the classical limit.

1. Introduction

The study of the adiabatic motion of charged particles in slowly-varying magnetic fields is a very well developed subject of classical mechanics [1]. It is of utmost importance in plasma physics, in the study of particle motion in the geomagnetic field as well as in astrophysics. For these reasons several efforts have been devoted over the years to gaining a deeper understanding of the formal structure underlying the problem. In this manner the original method of directly subjecting the equations of motion to a standard perturbative treatment [1] has been supported by the works of Gardner [2], Mynich [3] and Littlejohn [4–6] developing Hamiltonian theory and yielding a systematic procedure allowing the construction of adiabatic invariants. While Gardner and Mynich based their approach on field line coordinates and canonical transformations, Littlejohn proceeded by investigating the Hamiltonian structure of the system and by phase space (symplectic) geometrical techniques. In this paper we deal with the problem of the adiabatic motion of a charged, spinning, quantum particle in a slowly-varying magnetic field. For this task we develop an algebraic technique which very much reminds us of the one used by Littlejohn in classical mechanics. Nevertheless we make use of a different set of variables and do not make explicit use of symplectic geometry. We restrict our attention to particles moving in a plane under the influence of a time-independent perpendicular magnetic field. This is not only an expedient in order to illustrate our technique in a simple case. Although the planarity constraint may seem to be artificial in the classical context it is not in the quantum-mechanical context. Devices producing the confinement of electrons on a plane are in fact widely studied in physics, as, for example, in the quantum Hall effect [7]. When these systems interact with arbitrary inhomogeneous magnetic fields a realistic analysis indicates the effective dynamics to be described by a Hamiltonian proportional to the two-dimensional

Laplace operator in minimal coupling with an effective magnetic field represented by the normal component of the original magnetic field [8]. This is the kind of system which we are going to deal with. Our perturbative analysis is shown to be consistent with some general results concerning the ground state of quantum particles moving in arbitrary two-dimensional magnetic fields [9, 10].

Before facing the quantum problem it is better to briefly recall the main features of the classical one. We start therefore by considering a classical particle of mass m and charge e moving in a plane under the influence of a normal uniform magnetic field of magnitude B. Let $x = (x^1, x^2)$ denote the position of the particle and $v = (v^1, v^2)$ its velocity. As is well known the particle follows a circular orbit of radius $r_B = (mc/eB)|v|$ whose centre remains motionless. The frequency of the motion is the cyclotron frequency $\omega_B = eB/mc$. The centre of the orbit is called the *guiding centre* and its coordinates $X = (X^1, X^2)$ are related to the position and the velocity of the particle by the relation $X^i = x^i + (mc/eB)\varepsilon^{ik}v_k$†. Holding the energy of the particle fixed, the stronger the magnetic field the smaller is the radius of the orbit and the faster the particle. By increasing the magnitude B of the magnetic field, the classical particle explores a portion of the plane which becomes smaller and smaller. This gives a reason why, in classical mechanics, the motion in a very strong magnetic field may be studied along the same lines as that in a weakly inhomogeneous field. This fact is not immediate in quantum mechanics, since even in a uniform magnetic field the wavefunctions which are commonly used in the discussion of the problem do not localize in the neighbourhood of a point by increasing the magnetic field‡.
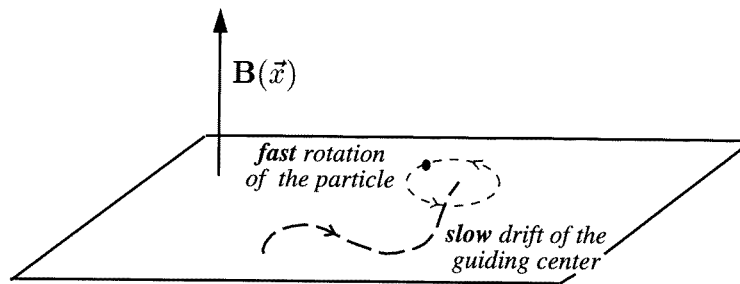

Figure 1. Adiabatic motion of a charged particle in a two-dimensional magnetic field.

Let us now consider the case of a weakly inhomogeneous magnetic field of magnitude $B(x)$. What happens is that the particle keeps on rapidly rotating around the guiding centre X but, this time, the guiding centre no longer remains motionless. It begins to slowly drift in the plane. The cyclotron frequency becomes, in addition, a function of the position $\omega_B(x) \simeq eB(X)/mc$. We can therefore distinguish two types of motion of the system: the very *fast* rotation around the guiding centre and the *slow* drift of the guiding centre in the plane. From the point of view of Hamiltonian theory it is not obvious to understand how to deal with this situation. The systematic procedure for finding a proper set of canonical variables for the system has been given by Gardner [2] whereas Littlejohn

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† $\varepsilon_{ij} = \varepsilon^{ij}$ denotes the completely antisymmetric tensor in two dimensions: $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$. Everywhere in this paper the sum over repeated indices is understood.

‡ Because of the infinite degeneracy of every Landau level, the energy eigenfunctions depend obviously on which variable we diagonalize along with the energy. It is customary to discuss the problem in the *Landau gauge* or in the *symmetric gauge* which lead, respectively, to the diagonalization of X^1 (X^2) and of the angular momentum of the particle L. In both cases the probability distributions of the particle position do not resemble the classical trajectories (in the ground state, for example, a point).

[4] has shown how to construct these variables as power series in an appropriate adiabatic parameter†. The rotational motion of the particle may then be separated from that of the guiding centre and the guiding centre motion described as the phase space motion of the remaining couple of canonical variables.

We now come to quantum mechanics. The first thing that strikes us is that Planck's constant $\hbar$ introduces a length scale into the problem: the *magnetic length*

$$l_B = \sqrt{\frac{\hbar c}{eB}}. \tag{1}$$

Quantum mechanics provides, therefore, a universal means to evaluate the order of magnitude of the variation of a magnetic field: the comparison between the length scale over which the field varies and the magnetic length. For a rather weak 1 G magnetic field, e is the electron charge, l_B is already a very small length, $l_B \simeq 10^{-4}$ cm. The magnetic length l_B appears therefore as a natural adiabatic parameter of the theory, that, in contrast to the classical case, does not have to be introduced by hand.

In this paper we construct a suitable set of operators allowing us to rewrite the Hamiltonian describing the motion of a particle in a two-dimensional magnetic field as a formal power series in the magnetic length l_B. With the expansion parameter having the dimension of length, the perturbative series makes sense only when the nth derivative of the field multiplied by $l_B{}^n$ is a very small number in every point of the plane.

In section 2 we give an outlook on the anomalous Hamiltonian structure of a particle interacting with a uniform magnetic field. The construction of a set of operators adapting to this simple problem indicates clearly the presence of a *fast* and a *slow* degree of freedom of the system. It also suggests the way to be followed in the general case. In section 3 we construct a new set of variables $\{\Pi_i, X^i; i = 1, 2\}$ which we adapt to the inhomogeneous problem. In the semiclassical picture the Π_is describe the rapid rotation of the particle, whereas the X^is take care of the dynamics of the guiding centre. The construction is carried out order by order in the perturbative parameter l_B up to second order. In section 4 the kinematical momenta π_i and the coordinates x^i, $i = 1, 2$, of the particle are rewritten in terms of the new variables. This allows us to write down the Hamiltonian of the system as a power series in the magnetic length l_B. An appropriate unitary transformation is finally used in section 5 in order to make the perturbative expansion depend on the variable Π_1 and Π_2 only by means of their combination $J = \frac{1}{2}(\Pi_1^2 + \Pi_2^2)$. The *fast* rotational motion of the particle may therefore be separated from the *slow* drift of the guiding centre. In the classical limit our result reproduces correctly Littlejohn's Hamiltonian [4]. Section 6 contains our conclusions.

2. An outlook on the canonical structure

Magnetic interactions appear as modifications of the canonical structure of dynamical systems. This feature plays a central role in the Hamiltonian, and hence quantum, description of the adiabatic motion of charged particles in strong magnetic fields. We begin, therefore, by briefly reviewing this aspect of the problem in the case of a homogeneous magnetic field, introducing at the same time conventions and notations that we shall use in the rest of the paper.

Let us consider a spin-1/2 particle of mass m, charge e and gyromagnetic factor g constrained to move in the x^1–x^2 plane in the presence of the homogeneous magnetic field

† In his work Littlejohn made use of a set of non-canonical variables. This may however be reconfigured to a canonical one by means of an appropriate transformation.

$B(x) = B\hat{z}$. Denoting by $a_i(x)$, $i = 1, 2$, an arbitrary choice of vector potential for the dimensionless field $B(x)/B$, $\partial_i a_j - \partial_j a_i = \varepsilon_{ij}$, the dynamics of the particle is described by the Pauli Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \sum_{i=1,2} \left(-i\hbar \frac{\partial}{\partial x^i} - \frac{eB}{c} a_i(x)\right)^2 + g\frac{\hbar eB}{mc}\sigma_3. \tag{2}$$

It is customary to parametrize the system by means of the canonical variables $p_i = -i\hbar\partial/\partial x^i$ and x^i, $i = 1, 2$. Nevertheless, Hamiltonian (2) takes an extremely simple form if we replace the *canonical momenta* p_i by the gauge covariant *kinematical momenta* π_i, $\pi_i = p_i - (eB/c)a_i(x)$. Apart from a scale factor, π_1 and π_2 behave as conjugate coordinates and the Hamiltonian (2) can be rewritten as that of an harmonic oscillator, allowing the immediate solution of the problem. In order to get an adequate set of canonical variables for the description of the system, it is therefore more convenient to keep the kinematical momenta π_i and to replace the coordinates x^i by the *guiding centre* operators X^i, $X^i = x^i + (c/eB)\varepsilon^{ij}\pi_j$. In the classical limit the X^is represent the coordinates of the centre of the classical orbit. The set of operators $\{\pi_i, X^i; i = 1, 2\}$ fulfils the commutation relations

$$[\pi_i, \pi_j] = i\frac{\hbar eB}{c}\varepsilon_{ij} \qquad [\pi_i, X^j] = 0 \qquad [X^i, X^j] = -i\frac{\hbar c}{eB}\varepsilon^{ij} \tag{3}$$

so that, up to some scale factors, π_i and X^i may be recognized as canonical variables. We note that the kinematical momentum π_1 and the guiding centre coordinates X^2 behave now as 'canonical coordinates' whereas the kinematical momentum π_2 and the guiding centre coordinate X^1 correspond to the respective 'canonical momenta'. The presence of a homogeneous magnetic field produces a kind of rotation of the canonical structure, mixing up canonical momenta and coordinates in new canonical variables.

An outlook on the canonical commutation relations (3) allows us to single out a second, very important, peculiarity of the variables π_i and X^i. Thinking of Heisenberg's equations of motion we immediately realize that the temporal variation of the π_is is proportional to B, $\dot{\pi}_i = i[\mathcal{H}, \pi_i]/\hbar \simeq B$, whereas that of the X^is goes as $1/B$, $\dot{X}^i = i[\mathcal{H}, X^i]/\hbar \simeq 1/B$†. In the limit of very strong magnetic fields, the canonical structure of the system is distorted from the magnetic scale B and the π_is and X^is behave naturally as describing a *fast* and a *slow* degree of freedom of the system, respectively.

In the next section we construct a set of operators generalizing the π_is and the X^is for an arbitrary inhomogeneous magnetic field $B(x) = Bb(x)\hat{z}$ (the dimension of the field being reabsorbed in the magnetic scale B). This allows us to set up an adiabatic description of the motion of a quantum particle in a two-dimensional magnetic field in a quite simple way.

For the sake of clarity it is convenient to introduce dimensionless quantities by factorizing the energy scale $\hbar\omega_B$, $\omega_B = eB/mc$, from the Hamiltonian. Recalling that the *magnetic length* $l_B = \sqrt{\hbar c/eB}$, we redefine the kinematical momenta as

$$\pi_i = -il_B \frac{\partial}{\partial x^i} - \frac{1}{l_B}a_i(x). \tag{4}$$

Hamiltonian (2) can then be written simply as $\mathcal{H} = \hbar\omega_B(\frac{1}{2}\sum\pi_i^2 - g\sigma_3)$. It is instead necessary to keep guiding centre operators with the dimension of length,

$$X^i = x^i + l_B\varepsilon^{ij}\pi_j \tag{5}$$

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† Hamiltonian (2), of course, does not depend on the X^is. We nevertheless act as if it does in order to explore the behaviour of the canonical variables in the limit of a strong magnetic field.

in order to preserve the scale dependence of the canonical commutation relations (3)

$$[\pi_i, \pi_j] = \mathrm{i}\varepsilon_{ij} \qquad [\pi_i, X^j] = 0 \qquad [X^i, X^j] = -\mathrm{i}l_B^2\varepsilon^{ij}. \tag{6}$$

A general inhomogeneous magnetic field $\boldsymbol{B}(\boldsymbol{x})$ depends on coordinates with the dimension of length, after all.

3. Looking for a suitable set of variables

We now consider a spin-1/2 particle of mass m, charge e and gyromagnetic factor g constrained to move on the x^1–x^2 plane under the action of the inhomogeneous magnetic field $\boldsymbol{B}(\boldsymbol{x}) = Bb(\boldsymbol{x})\hat{\boldsymbol{z}}$. $b(\boldsymbol{x})$ is supposed to be an arbitrary positive nevervanishing smooth function of $\boldsymbol{x}$. Introducing an arbitrary choice of vector potential $a_i(\boldsymbol{x})$ for the dimensionless field $b(\boldsymbol{x})\hat{\boldsymbol{z}}$, $\partial_i a_j - \partial_j a_i = \varepsilon_{ij} b(\boldsymbol{x})$, the system is again described by the Pauli Hamiltonian

$$\mathcal{H} = \hbar\omega_B\left(\tfrac{1}{2}\sum_{i=1,2}\pi_i^2 + g\sigma_3 b(\boldsymbol{x})\right) \tag{7}$$

where the kinematical momenta $\pi_i = -\mathrm{i}l_B\partial_i - a_i(\boldsymbol{x})/l_B$ have already been introduced in place of the canonical momenta p_i. In spite of the very simple dependence of Hamiltonian (7) on the π_is, the kinematical momenta are no longer conjugate variables and cannot be directly used to give a simple solution to the problem. The set of operators $\{\pi_i, x^i; i = 1, 2\}$ fulfils in fact the commutation relations

$$[\pi_i, \pi_j] = \mathrm{i}\varepsilon_{ij} b(\boldsymbol{x}) \qquad [\pi_i, x^j] = -\mathrm{i}l_B\delta_i^j \qquad [x^i, x^j] = 0. \tag{8}$$

Nevertheless, we can try to construct a set of variables generalizing the kinematical momenta (4) and the guiding centre operators (5) to the case of an inhomogeneous magnetic field as power series in the magnetic length l_B with coefficients depending on the π_is and the x^is. We proceed order by order in the parameter l_B by constructing, say at the nth order, a set of operators $\{\Pi_i^{(n)}, X_{(n)}^i; i = 1, 2\}$ fulfilling adequate conditions. We require that:

  • in the limit of a constant magnetic field, $b(x) \to 1$, the $\Pi_i^{(n)}$s and the $X_{(n)}^i$s should reduce to the kinematical momenta (4) and to the guiding centre operators (5), respectively;
  • the $\Pi_i^{(n)}$s should be conjugate variables up to terms of order $l_B{}^n$; and
  • the $X_{(n)}^i$s should commute with the $\Pi_i^{(n)}$s up to terms of order $l_B{}^n$.

These commutation relations obviously generalize equation (6). We nevertheless do not insist that the $X_{(n)}^i$s be conjugate variables [4].

3.1. Zero-order variables

In order to fulfil the conditions above up to terms of order l_B we simply rescale the kinematical momenta π_i by a factor $b^{-1/2}$,

$$\Pi_i^{(0)} = \tfrac{1}{2}\{b^{-1/2}, \pi_i\} \tag{9}$$

where the function $b^{-1/2}$ should be evaluated in $\boldsymbol{X}_{(0)} \equiv \boldsymbol{x}$. The anticommutator $\{,\}$ is introduced in order to make the $\Pi_i^{(0)}$s Hermitian. A brief computation gives the commutation relation fulfilled by the set of operators $\{\Pi_i^{(0)}, X_{(0)}^i; i = 1, 2\}$,

$$[\Pi_i^{(0)}, \Pi_j^{(0)}] = \mathrm{i}\varepsilon_{ij} + \mathrm{i}\varepsilon_{ij}\frac{l_B}{4}\varepsilon^{kl}\left\{\frac{(\partial_k b)}{b^{3/2}}, \Pi_l^{(0)}\right\} \qquad [\Pi_i^{(0)}, X_{(0)}^j] = -\mathrm{i}l_B\delta_i^j b^{-1/2}$$

$$[X_{(0)}^i, X_{(0)}^j] = 0 \tag{10}$$

where all the functions have to be evaluated in $X_{(0)}$. As required, the $\Pi_i^{(0)}$s are conjugate variables up to terms of order l_B and the commutators between the $\Pi_i^{(0)}$s and the $X_{(0)}^i$s are again of order l_B.

3.2. First-order variables

We proceed by constructing a couple of operators $X_{(1)}^i$, $i = 1, 2$, commuting with the $\Pi_i^{(0)}$s up to terms of order $l_B{}^2$. The goal is achieved by performing a transformation generalizing (5) to the case of an inhomogeneous magnetic field. $X_{(1)}^i$ is defined as

$$X_{(1)}^i = x^i + \frac{l_B}{2} \varepsilon^{ik} \{b^{-1/2}, \Pi_k^{(0)}\} \tag{11}$$

the function $b^{-1/2}$ being evaluated in $X_{(0)}$. We have, therefore, the following set of commutation relations:

$$[\Pi_i^{(0)}, \Pi_j^{(0)}] = i\varepsilon_{ij} + i\varepsilon_{ij} \frac{l_B}{4} \varepsilon^{km} \left\{ \frac{(\partial_k b)}{b^{3/2}}, \Pi_m^{(0)} \right\}$$

$$+ - i\varepsilon_{ij} \frac{l_B{}^2}{8} \varepsilon^{km} \varepsilon^{ln} \left\{ \frac{(\partial_k \partial_l b)}{b^2} - \frac{3}{2} \frac{(\partial_k b)(\partial_l b)}{b^3}, \{\Pi_m^{(0)}, \Pi_n^{(0)}\} \right\} + o(l_B{}^3)$$

$$[\Pi_i^{(0)}, X_{(1)}^j] = i l_B{}^2 \varepsilon^{jk} \left(\frac{1}{2} \left\{ \frac{(\partial_i b)}{b^2}, \Pi_k^{(0)} \right\} - \frac{1}{4} \left\{ \frac{(\partial_k b)}{b^2}, \Pi_i^{(0)} \right\} \right)$$

$$[X_{(1)}^i, X_{(1)}^j] = -i l_B{}^2 \varepsilon^{ij} b^{-1} + o(l_B{}^3). \tag{12}$$

All the functions have to be evaluated in $X_{(1)}$. We observe that the $X_{(1)}^i$s are no longer commuting variables so that the evaluation of the magnetic field function b and of its derivatives in $X_{(1)}$ involves ordering ambiguities. With the commutator of the $X_{(1)}^i$ being of order $l_B{}^2$, these ambiguities concern the terms $o(l_B{}^3)$.

The operators $\Pi_i^{(1)}$s are constructed by adding a correction of order l_B to the $\Pi_i^{(0)}$s. This correction is looked for as a homogeneous second-order polynomial in the $\Pi_i^{(0)}$s. A brief computation indicates that the right choice is

$$\Pi_i^{(1)} = \Pi_i^{(0)} - \frac{l_B}{24} \varepsilon^{km} \left\{ \frac{(\partial_k b)}{b^{3/2}}, \{\Pi_i^{(0)}, \Pi_m^{(0)}\} \right\} \tag{13}$$

where the functions are evaluated in $X_{(1)}$. The commutation relations fulfilled by the set of operators $\{\Pi_i^{(1)}, X_{(1)}^i; i = 1, 2\}$ can be immediately obtained by means of (12):

$$[\Pi_i^{(1)}, \Pi_j^{(1)}] = i\varepsilon_{ij} - i\varepsilon_{ij} \frac{l_B{}^2}{4} \varepsilon^{km} \varepsilon^{ln} \left\{ \frac{1}{2} \frac{(\partial_k \partial_l b)}{b^2} - \frac{5}{9} \frac{(\partial_k b)(\partial_l b)}{b^3}, \{\Pi_m^{(1)}, \Pi_n^{(1)}\} \right\} + o(l_B{}^3)$$

$$[\Pi_i^{(1)}, X_{(1)}^j] = i l_B{}^2 \varepsilon^{jk} \left(\frac{1}{2} \left\{ \frac{(\partial_i b)}{b^2}, \Pi_k^{(1)} \right\} - \frac{1}{4} \left\{ \frac{(\partial_k b)}{b^2}, \Pi_i^{(1)} \right\} \right)$$

$$[X_{(1)}^i, X_{(1)}^j] = -i l_B{}^2 \varepsilon^{ij} b^{-1} + o(l_B{}^3). \tag{14}$$

The $\Pi_i^{(1)}$s behave as conjugate variables up to terms of order $l_B{}^2$ and the commutators between the $\Pi_i^{(1)}$s and the $X_{(1)}^i$s vanish again up to second-order terms in l_B.

3.3. Second-order variables

The second- and higher-order variables have to be constructed along the same lines as the previous ones. Looking for the $l_B{}^2$ correction to be added to the $X^i_{(1)}$s as a homogeneous second-order polynomial in the $\Pi^{(1)}_i$s, we easily find

$$X^i_{(2)} = X^i_{(1)} - \frac{l_B{}^2}{8} \varepsilon^{ik} \varepsilon^{jl} \left\{ \frac{(\partial_j b)}{b^2}, \left\{ \Pi^{(1)}_k, \Pi^{(1)}_l \right\} \right\} \tag{15}$$

the functions being evaluated in $X_{(1)}$. Relations (14) allow us to compute the commutation relations

$$[\Pi^{(1)}_i, \Pi^{(1)}_j] = \mathrm{i}\varepsilon_{ij} - \mathrm{i}\varepsilon_{ij} \frac{l_B{}^2}{4} \varepsilon^{km} \varepsilon^{ln} \left\{ \frac{1}{2} \frac{(\partial_k \partial_l b)}{b^2} - \frac{5}{9} \frac{(\partial_k b)(\partial_l b)}{b^3}, \{\Pi^{(1)}_m, \Pi^{(1)}_n\} \right\} + o(l_B{}^3)$$

$$[\Pi^{(1)}_i, X^j_{(2)}] = 0 + o(l_B{}^3)$$

$$[X^i_{(2)}, X^j_{(2)}] = -\mathrm{i}l_B{}^2 \varepsilon^{ij} b^{-1} + o(l_B{}^3) \tag{16}$$

where every function is now evaluated in $X_{(2)}$. Finally we proceed by constructing $\Pi^{(2)}_i$ by adding an adequate homogeneous third-order polynomial in the $\Pi^{(1)}_i$s to $\Pi^{(1)}_i$,

$$\Pi^{(2)}_i = \Pi^{(1)}_i + \frac{l_B{}^2}{4} \varepsilon^{km} \varepsilon^{ln} \left\{ \frac{1}{2} \frac{(\partial_k \partial_l b)}{b^2} - \frac{5}{9} \frac{(\partial_k b)(\partial_l b)}{b^3}, \Pi^{(1)}_m \Pi^{(1)}_i \Pi^{(1)}_n \right\} \tag{17}$$

the function b and its derivatives again being evaluated in $X_{(2)}$. The set of operators $\{\Pi^{(2)}_i, X^i_{(2)}; i = 1, 2\}$ fulfils the desired commutation relations

$$[\Pi^{(2)}_i, \Pi^{(2)}_j] = \mathrm{i}\varepsilon_{ij} + o(l_B{}^3) \qquad [\Pi^{(2)}_i, X^j_{(2)}] = 0 + o(l_B{}^3)$$

$$[X^i_{(2)}, X^j_{(2)}] = -\mathrm{i}l_B{}^2 \varepsilon^{ij} b^{-1} + o(l_B{}^3). \tag{18}$$

The $\Pi^{(2)}_i$s are conjugate up to terms of order $l_B{}^3$ and commutate with the $X^i_{(2)}$s again up to third order in l_B. In the limit of a constant magnetic field, of course, all the variables that we have introduced have the correct behaviour.

3.4. A non-canonical set of variables

In principle we may think to repeat this procedure an arbitrary number of times and to construct, as power series in l_B with coefficients polynomial in the π_is and depending on the x^is through the function b and its derivatives, a set of non-canonical operators $\{\Pi_i, X^i; i = 1, 2\}$ fulfilling the commutation relations

$$[\Pi_i, \Pi_j] = \mathrm{i}\varepsilon_{ij} \qquad [\Pi_i, X^j] = 0 \qquad [X^i, X^j] = -\mathrm{i}l_B{}^2 \varepsilon^{ij} b^{-1} \tag{19}$$

b^{-1} now being evaluated in X. However, in order to discuss the problem to second order in the adiabatic parameter l_B, we only need to know the first three terms of these series, which are terms which we have already evaluated. Up to terms of order $l_B{}^3$ we may therefore confuse the $\Pi^{(2)}_i$s and $X^i_{(2)}$s with the Π_is and X^is, respectively.

Let us observe now, that aside from the function b^{-1} appearing on the right-hand side of the third identity, the commutation relations (19) correspond to (6). Although the X^is are not conjugate variables they commute with the Π_is and their commutator is a function of the X^is alone. The Hilbert space of the system separates therefore under the action of the two pairs of operators in the direct sum of the two subspaces, each one describing a degree of freedom of the system. The scale dependence of the

commutation relations (19) indicates again that in the limit of very strong magnetic fields (small l_B) the Π_is and the X^is describe a *fast* and a *slow* degree of freedom of the system, respectively. The *adiabatic kinematical momenta* Π_i and the *adiabatic guiding centre operators* X^i, introduced in this section, appear therefore as a suitable choice of variables for our problem.

4. The adiabatic expansion

The next step is to rewrite Hamiltonian (7) in terms of the new set of variables. For this purpose we have first to invert the power series expressing the Π_is and the X^is in terms of the π_is and the x^is and then to replace these expressions in (7). As a result the Hamiltonian will appear as a power series in the magnetic length l_B:

$$\mathcal{H} = \mathcal{H}^{(0)} + l_B \mathcal{H}^{(1)} + l_B{}^2 \mathcal{H}^{(2)} + \cdots. \tag{20}$$

In the limit of the slowly-varying magnetic field, that is when the magnetic length may be considered small with respect to the length scale over which the magnetic field varies, this equation may be interpreted as a perturbative expansion of the Hamiltonian and used to obtain approximate expressions of the spectrum and the wavefunctions of the system. We call it the *adiabatic expansion* of the Hamiltonian.

The task of obtaining the operators π_i and x^i in terms of the new variables Π_i and X^i is not a hard one. Recalling that $X^i = X^i_{(2)} + o(l_B{}^3)$ and using equations (9), (11), (13) and (15) we can immediately obtain the first three terms of the power series expressing the x^is in terms of the π_is and the X^is, $x^i = x^i(\pi, X)$. This allows us to rewrite the $\Pi_i^{(0)}$s as functions of the the π_is and the X^is, so that by using equations (13), (17) and the relation $\Pi_i = \Pi_i^{(2)} + o(l_B{}^3)$, it is possible to rewrite the Π_is as a power series in l_B with coefficients depending on the π_is and the X^is, $\Pi_i = \Pi_i(\pi, X)$. Inverting these series order by order we can finally calculate the π_is as functions of the Π_is and the X^is. By substituting these expressions in $x^i = x^i(\pi, X)$, we also get the x^is as functions of the Π_is and the X^is. A few computations lead to the result

$$\pi_i = b^{1/2}\Pi_i - \frac{l_B}{6}\frac{(\partial_k b)}{b}\varepsilon^{km}\{\Pi_i, \Pi_m\} + l_B{}^2 \left(\frac{1}{8}\frac{(\partial_k \partial_l b)}{b^{3/2}} - \frac{7}{72}\frac{(\partial_k b)(\partial_l b)}{b^{5/2}}\right)\varepsilon^{km}\varepsilon^{ln}\Pi_m\Pi_i\Pi_n$$
$$+ o(l_B{}^3) \tag{21}$$

$$x^i = X^i - l_B\varepsilon^{ik}b^{-1/2}\Pi_k - \frac{l_B{}^2}{12}\varepsilon^{ik}\varepsilon^{jl}\frac{(\partial_j b)}{b^2}\{\Pi_k, \Pi_l\} + o(l_B{}^3) \tag{22}$$

the function b and its derivatives being evaluated in X. As a check we can easily reobtain the commutation relations (8) by means of expressions (21), (22) and (19).

The first three terms of the adiabatic expansion of the Hamiltonian can now be immediately obtained by substituting expressions (21) and (22) in (7). In order to simplify the notation we introduce the operators $J = \frac{1}{2}\delta^{ij}\Pi_i\Pi_j$, $J_k = \frac{1}{2}\delta^{ij}\Pi_i\Pi_k\Pi_j$ and $J_{kl} = \frac{1}{2}\delta^{ij}\Pi_i\{\Pi_k, \Pi_l\}\Pi_j$. J is just the Hamiltonian of an harmonic oscillator in the conjugate variables Π_1, Π_2 and, as we will see in the next section, plays a central role in our discussion. A brief computation yields

$$\mathcal{H}^{(0)}/\hbar\omega_B = b(J + g\sigma_3) \tag{23}$$

$$\mathcal{H}^{(1)}/\hbar\omega_B = -\frac{(\partial_i b)}{b^{1/2}}\varepsilon^{ik}(2J_k/3 + g\sigma_3\Pi_k) \tag{24}$$

$$\mathcal{H}^{(2)}/\hbar\omega_B = \left(\frac{1}{8}\frac{(\partial_i\partial_j b)}{b} - \frac{3}{72}\frac{(\partial_i b)(\partial_j b)}{b^2}\right)\varepsilon^{ik}\varepsilon^{jl}(J_{kl} + 2g\sigma_3\{\Pi_k, \Pi_l\})$$
$$-\left(\frac{1}{16}\frac{(\partial_i\partial_j b)}{b} - \frac{5}{144}\frac{(\partial_i b)(\partial_j b)}{b^2}\right)\delta^{ij} \tag{25}$$
$$\vdots$$

These equations represent a good starting point for a perturbative analysis of the system. Nevertheless there is still some more work that may be done in order to make this task simpler. In the next section we will follow a strategy which very much recalls the one used in the perturbation theory of Hamiltonian mechanics, in order to find a unitary transformation making $\mathcal{H}$ depend on Π_1 and Π_2 only by means of their combination J and its powers. J is then an adiabatic invariant which may be identified with the *magnetic moment of gyration* of the particle.

5. Effective guiding centre dynamics

A well known strategy in treating perturbative problems in classical mechanics is that of subjecting the system to a series of near-identity canonical transformations in order to make the various orders of the perturbative expansion of the Hamiltonian be independent of certain variables. In this way it is possible to identify the constants of motion of the system up to an arbitrary order in the perturbative parameter. The approximate solution of the problem turns out to be notably simplified. In this section we will adopt the equivalent technique in quantum mechanics. Hamiltonian (20) will be subjected to a near-identity unitary transformation such that the following criterion is fulfilled: *the new Hamiltonian should depend on the conjugate variables Π_1 and Π_2 only by means of the operator J and its powers*. In this way J is a constant of motion of the system and the dynamics of the X^is separates from that of the Π_is.

Let us therefore consider a near-identity unitary operator U, represented in the form $U = \mathrm{e}^{\mathrm{i}\mathcal{L}}$. The Hermitian operator $\mathcal{L}$ will differ from the identity operator $\mathbf{1}$ by terms of order greater than l_B so that it may be represented as the power series

$$\mathcal{L} = \mathbf{1} + l_B\mathcal{L}^{(1)} + l_B{}^2\mathcal{L}^{(2)} + \cdots. \tag{26}$$

Subjecting Hamiltonian (20) to this transformation produces the result

$$\mathrm{e}^{\mathrm{i}\mathcal{L}}\mathcal{H}\mathrm{e}^{-\mathrm{i}\mathcal{L}} = \mathcal{H}^{(0)} + l_B(\mathcal{H}^{(1)} + \mathrm{i}[\mathcal{L}^{(1)}, \mathcal{H}^{(0)}])$$
$$+ l_B{}^2\left(\mathcal{H}^{(2)} + \mathrm{i}[\mathcal{L}^{(2)}, \mathcal{H}^{(0)}] + \mathrm{i}\left[\mathcal{L}^{(1)}, \mathcal{H}^{(1)} + \frac{\mathrm{i}}{2}[\mathcal{L}^{(1)}, \mathcal{H}^{(0)}]\right]\right) + \cdots. \tag{27}$$

The zero-order term of the Hamiltonian remains unchanged whereas the others are corrected by additive terms depending on the commutators of the $\mathcal{L}^{(n)}$s with $\mathcal{H}^{(0)}$ and the subsequent terms of the adiabatic expansion. We start therefore by looking for an operator $\mathcal{L}^{(1)}$ such that its commutator with $\mathcal{H}^{(0)}$, $[\mathcal{L}^{(1)}, \mathcal{H}^{(0)}]$, might correct the first-order term of the adiabatic expansion in the desired way. A brief computation indicates that it is possible to annihilate the first-order term of the transformed Hamiltonian by choosing

$$\mathcal{L}^{(1)} = -\frac{(\partial_i b)}{b^{3/2}}\delta^{ik}(2J_k/3 + g\sigma_3\Pi_k). \tag{28}$$

The next step is to construct the operator $\mathcal{L}^{(2)}$ making the second-order term of the adiabatic expansion, $\mathcal{H}^{(2)} + \mathrm{i}[\mathcal{L}^{(2)}, \mathcal{H}^{(0)}] + \mathrm{i}[\mathcal{L}^{(1)}, \mathcal{H}^{(1)}]/2$, depend on Π_1, Π_2 only by means of their

combination J and its powers. The correct choice results in

$$\mathcal{L}^{(2)} = \frac{1}{16} \left(\frac{(\partial_i \partial_j b)}{b^2} - \frac{1}{3} \frac{(\partial_i b)(\partial_j b)}{b^3} \right) \varepsilon^{ik} \delta^{jl} (J_{kl} + 2g\sigma_3 \{\Pi_k, \Pi_l\}). \qquad (29)$$

Having obtained further terms of the adiabatic expansion (20) it would be possible to continue with this procedure to arbitrary order. At least in principle therefore it is possible to make J into a constant of motion to an arbitrary order in the adiabatic parameter l_B. Our analysis stops at second order. Subjecting Hamiltonian (20), with $\mathcal{H}^{(0)}$, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ given by (23), (24) and (25), respectively, to the unitary transformation already described we get the new adiabatic expansion

$$\mathcal{H}'/\hbar\omega_B = b(J + g\sigma_3) + \frac{l_B^2}{4} \left[\frac{\triangle b}{b} - 3\frac{|\nabla b|^2}{b^2} \right] (J^2 + 2g\sigma_3 J)$$
$$+ \frac{l_B^2}{16} \left[\frac{\triangle b}{b} - (1 + 8g^2)\frac{|\nabla b|^2}{b^2} \right] + o(l_B^3) \qquad (30)$$

where the functions b, $\triangle b = \delta^{ij}(\partial_i \partial_j b)$ and $|\nabla b|^2 = \delta^{ij}(\partial_i b)(\partial_j b)$ are again evaluated in $\boldsymbol{X}$. Up to terms of order l_B^3 the operator J is a constant of motion of the system and the harmonic oscillator degree of freedom described by Π_1 and Π_2 separates from the the guiding centre motion described by X^1 and X^2. In the semiclassical picture the fast rotation of the particle around the guiding centre corresponds to the phase space motion in the plane Π_1–Π_2 whereas the slow drift of the guiding centre in the plane corresponds to the phase space motion in the plane X^1–X^2. The adiabatic invariant J may therefore be identified with the *magnetic moment of gyration* of the particle, that is the magnetic moment of the current loop described by the particle in one cyclotron gyration. In the classical limit the Hamiltonian operator (30) reproduces correctly Littlejohn's guiding centre Hamiltonian [4].

Once the system has been frozen in one of its gyrating eigenstates, that is, in an eigenstate of the *fast* degree of freedom, Hamiltonian (30) describes the corresponding effective guiding centre dynamics. It is interesting to observe that the zero-order effective guiding centre Hamiltonian corresponds to the magnetic field function $b(\boldsymbol{x})$ in which the coordinates x^1 and x^2 have been substituted by the non-commuting operators X^1 and X^2. Introducing a pair of Euler potentials $\mathrm{x}^1(\boldsymbol{x})$ and $\mathrm{x}^2(\boldsymbol{x})$ for the magnetic field $b(\boldsymbol{x})\hat{\boldsymbol{z}}$, $\nabla \mathrm{x}^1 \wedge \nabla \mathrm{x}^2 = b$ [11], it is also possible to make the X^is into a couple of conjugate operators. Defining $X^1 = \mathrm{x}^1(\boldsymbol{X})$ and $X^2 = \mathrm{x}^2(\boldsymbol{X})$ we get in fact $[X^1, X^2] = -il_B^2$ [2,4]. In order to evaluate the zero-order spectrum of the system it is therefore sufficient to construct the Hamiltonian operator of the system by substituting the commuting variables x^1 and x^2 with the couple of conjugate operators X^1 and X^2 in the function $b(\boldsymbol{x}(\mathrm{x}^1, \mathrm{x}^2))$. This is similar to a quantization procedure, which for some aspects, reminds us of the one explored by Klauder and Onofri [12, 13].

Further corrections to the spectrum may be evaluated by means of standard perturbation theory and the subsequent terms of the adiabatic expansion (30).

To conclude this section we specialize to the case of electrons, $g = -1/2$, in order to check that our results are consistent with a theorem of Aharonov and Casher [9] stating, among other things, that the ground state of an electron moving in an arbitrary two-dimensional magnetic field always has zero energy. By setting the gyromagnetic factor to $-1/2$ our Hamiltonian reduces to the quite compact form

$$\mathcal{H}'/\hbar\omega_B = b(J - \tfrac{1}{2}\sigma_3) + \frac{l_B^2}{4} \left[\frac{\triangle b}{b} - 3\frac{|\nabla b|^2}{b^2} \right] (J^2 - \sigma_3 J + \tfrac{1}{4}) + o(l_B^3). \qquad (31)$$

Recalling that the eigenvalues of J are $E_n = (n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$, and that of σ_3 are $s = \pm 1$ it can immediately be observed that the ground-state energy of Hamiltonian (31), ($n = 0$, $s = 1$), is zero up to terms of order $l_B{}^3$.

6. Discussion and conclusion

The construction of a set of non-canonical operators sharing the main properties of the kinematical momenta and the guiding centre operators of a particle moving in a homogeneous magnetic field allows us to separate the unperturbed dynamics of a quantum particle moving in a two-dimensional weakly-inhomogeneous magnetic field from the perturbation, making the guiding centre dynamics non-trivial. The effective guiding centre Hamiltonian can be written furthermore as a power series in the magnetic length l_B. The main features of the construction are that it is perturbative in nature, that is, it is performed order by order in the adiabatic parameter l_B, and involves only simple algebraic manipulations. These properties allow us to achieve our goal in a very simple and economical way.

It is worthwhile saying something about the efficiency of the adiabatic expansion (30). As we said in the introduction, the presence of a magnetic field in a quantum context introduces the length scale l_B. Intuitively, we may therefore consider a magnetic field to be slowly varying (weakly inhomogeneous), when its variation ratio over the magnetic length scale l_B is a small number. The adiabatic expansion obtained in this paper completely confirms this picture. Furthermore, for a rather weak magnetic field of 1 G, the magnetic length is still very small, $l_B \simeq 10^{-4}$ cm. To have an idea of some number, we recall that the geomagnetic field near to the surface of the earth is of the order of a gauss as well as the magnetic field produced by a wire in which flows a current of a few amperes. In the physics of the quantum Hall effect much stronger fields are used, typically of the order of 10^4–10^5 G. All these fields vary significantly over lengths which go from millimetres to several metres. That is to say, the adiabatic approximation is a *very* good one. Among other things, therefore, our work gives a further explanation of why such sharp Landau levels are observed in the quantum Hall effect.

Once the system has been frozen in one of its gyrating eigenstates, the dynamics of the remaining degree of freedom is described by means of a Hamiltonian operator which has to be constructed by quantizing the classical Hamiltonian $b(\boldsymbol{x}(\mathrm{x}^1, \mathrm{x}^2))$ (replacing $\hbar$ with $l_B{}^2$). The $l_B{}^2$ terms of the adiabatic expansion (30) will in general contribute a small correction to the spectrum of the system. By an appropriate choice of the magnetic field, therefore, an arbitrary one-degree-of-freedom Hamiltonian may be reproduced. It would be interesting to explore the possibility of concretely realizing devices of this kind.

We conclude by observing that the method we developed in this paper works in the full three-dimensional case as well as in the two-dimensional one. The study of the adiabatic motion of a charged, spinning, quantum particle in a three-dimensional magnetic field will be reported upon in a forthcoming publication.

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